九十六學年第一學期 PHYS2310 電磁學 期中考試題(共兩頁)

[Griffiths Ch.1-3]　2007/11/13,　10:10am–12:00am, 教師：張存績

記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

◇ Useful formulas

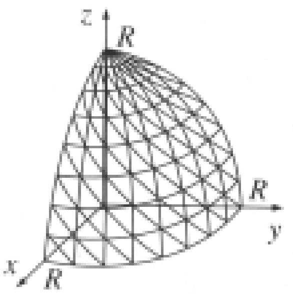
$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\varphi}} \quad \text{and} \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}v_\phi$$

1. (8%,12%) $\mathbf{v} = r^2\cos\theta\hat{\mathbf{r}} + r^2\cos\phi\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\hat{\boldsymbol{\varphi}}$

   (a) Compute $\nabla \cdot \mathbf{v}$.

   (b) Check the divergence theorem using the volume shown in the figure (one octant of the sphere of radius R).

   [Hint: Make sure you include the entire surface.]



2. (6%,7%,7%) A solid sphere of radius R is uniformly charged with the volume charge density ρ.

   (a) Find the voltage V and the electric field $\mathbf{E}$ in the two regions $r \le R$ and $r > R$.

   (b) Find the total energy using ρ and V.

   (c) Find the total energy using $\mathbf{E}$ and V. Take a spherical volume of radius a ($a > R$).

   [Hint: The energy $W = \frac{1}{2}\int \rho V d\tau = \frac{\varepsilon_0}{2}\left(\int_V E^2 d\tau + \oint_S V\mathbf{E}\cdot d\mathbf{a}\right)$.]

3. (10%, 10%) A metal sphere of radius R carries a total charge Q.

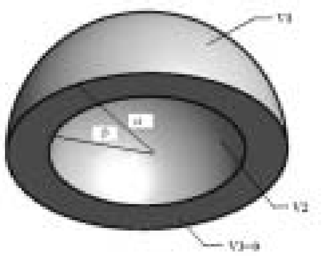
   (a) Use the boundary condition to determine the surface charge density σ on the metal sphere ($r = R$).

   (b) Calculate the force of repulsion between the "northern" hemisphere and the "southern" hemisphere.

   [Hint: The repulsive force per unit area (or the electrostatic pressure) $\mathbf{f} = \frac{\varepsilon_0}{2}E^2\hat{\mathbf{n}}$.]

4. (7%,7%,6%) A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane.
   (a) Find the potential V in the region above the plane.
   (b) Find the surface charge density σ induced on the conducting plane.
   (c) Find the force on the wire per unit length.
   [Hint: Use the method of images.]

5. (6%, 8%, 6%) Suppose the potential at the surface of a hollow hemisphere is specified, as shown in the figure, where $V_1(a,\theta) = V_0(5\cos^3\theta - 3\cos\theta)$, $V_2(b,\theta) = 0$, $V_3(r,\pi/2) = 0$. V_0 is a constant.
   (a) Show the general solution in the region $b \le r \le a$.
   (b) Determine the potential in the region $b \le r \le a$, using the boundary conditions.
   (c) Calculate the electric field at the inner shell $\mathbf{E}(r=b)$ and outer shell $\mathbf{E}(r=a)$.
   [Hint: $P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2 - 1)/2,$ and $P_3(x) = (5x^3 - 3x)/2$.]



1.

(a)

$$\mathbf{v} = r^2\cos\theta\,\hat{\mathbf{r}} + r^2\cos\phi\,\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}}$$

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}v_\phi \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta r^2\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi) \\ &= 4r\cos\theta + r\frac{\cos\theta}{\sin\theta}\cos\phi - r\frac{\cos\theta}{\sin\theta}\cos\phi \\ &= 4r\cos\theta\end{aligned}$$

(b)

The divergence theorem $\int_V \nabla\cdot\mathbf{v}\,d\tau = \oint_S \mathbf{v}\cdot d\mathbf{a}$

$$\begin{aligned}\int_V \nabla\cdot\mathbf{v}\,d\tau &= \int_0^R\int_0^{\pi/2}\int_0^{\pi/2}(4r\cos\theta)r^2\sin\theta\,dr\,d\theta\,d\phi = \frac{\pi}{2}\int_0^R\int_0^{\pi/2}(4r^3\cos\theta\sin\theta)\,dr\,d\theta \\ &= \frac{\pi}{16}\int_0^R\int_0^{\pi/2}(4\sin 2\theta)\,dr^4\,d\theta = \frac{\pi}{4}R^4\int_0^{\pi/2}\sin 2\theta\,d2\theta = \frac{\pi}{4}R^4\end{aligned}$$

$$\oint_S \mathbf{v}\cdot d\mathbf{a} = xy\text{-plane} + yz\text{-plane} + zx\text{-plane} + \text{curved surface}$$

xy-plane: $d\mathbf{a} = -r\,dr\,d\theta\,\hat{\boldsymbol{\phi}},\ \phi = 0,\ \mathbf{v}\cdot d\mathbf{a} = (r^2\cos\theta\sin\phi)r\,dr\,d\theta = 0,$

yz-plane: $d\mathbf{a} = r\,dr\,d\theta\,\hat{\boldsymbol{\phi}},\ \phi = \pi/2,\ \mathbf{v}\cdot d\mathbf{a} = -(r^2\cos\theta\sin\phi)r\,dr\,d\theta = -r^3\cos\theta\,dr\,d\theta,\ \int_0^R\int_0^{\pi/2} -(r^3\cos\theta)\,dr\,d\theta = -\frac{1}{4}R^4$

zx-plane: $d\mathbf{a} = r\,dr\,d\phi\,\hat{\boldsymbol{\theta}},\ \theta = \pi/2,\ \mathbf{v}\cdot d\mathbf{a} = (r^2\cos\phi)r\,dr\,d\phi = r^3\cos\phi\,dr\,d\phi,\ \int_0^R\int_0^{\pi/2}(r^3\cos\phi)\,dr\,d\phi = \frac{1}{4}R^4$

curved surface: $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}},\ r = R,\ \mathbf{v}\cdot d\mathbf{a} = (R^2\cos\theta)R^2\sin\theta\,d\theta\,d\phi = \frac{R^4}{2}\sin 2\theta\,d\theta\,d\phi,$

$$\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{R^4}{2}\sin 2\theta\right)d\theta\,d\phi = \frac{\pi}{4}R^4$$

$$\oint_S \mathbf{v}\cdot d\mathbf{a} = 0 - \frac{1}{4}R^4 + \frac{1}{4}R^4 + \frac{\pi}{4}R^4 = \frac{\pi}{4}R^4 = \int_V \nabla\cdot\mathbf{v}\,d\tau$$

2.

(a)

Use Gauss's law, we obtain $E = \begin{cases} \dfrac{1}{4\pi\varepsilon_0 r^2}\dfrac{4}{3}\pi\rho r^3 = \dfrac{\rho}{3\varepsilon_0}r, & \text{for } r \le R \\ \dfrac{1}{4\pi\varepsilon_0 r^2}\dfrac{4}{3}\pi\rho R^3 = \dfrac{\rho R^3}{3\varepsilon_0}\dfrac{1}{r^2}, & \text{for } r \ge R \end{cases}$ in $\hat{\mathbf{r}}$

$$V = \begin{cases} -\int_\infty^R \mathbf{E}\cdot d\mathbf{l} - \int_R^r \mathbf{E}\cdot d\mathbf{l} = \dfrac{\rho R^3}{3\varepsilon_0}\dfrac{1}{R} + \dfrac{\rho}{6\varepsilon_0}(R^2 - r^2) = \dfrac{\rho}{6\varepsilon_0}(3R^2 - r^2), & \text{for } r \le R \\ -\int_\infty^r \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^r \dfrac{\rho R^3}{3\varepsilon_0}\dfrac{1}{r^2}\cdot dr = \dfrac{\rho R^3}{3\varepsilon_0}\dfrac{1}{r}, & \text{for } r \ge R \end{cases}$$

(b) $W = \frac{1}{2}\int \rho V d\tau = \frac{\rho}{2}\int_0^R V d\tau$

$$W = \frac{1}{2}\int \rho V d\tau = \frac{\rho}{2}\int_0^R \left(\frac{\rho}{6\varepsilon_0}(3R^2 - r^2)\right) r^2 dr \iint \sin\theta d\theta d\phi$$
$$= \frac{\rho}{2}\frac{\rho}{6\varepsilon_0}\left(3R^2\frac{R^3}{3} - \frac{R^5}{5}\right)4\pi$$
$$= \frac{4\pi\rho^2 R^5}{15\varepsilon_0}$$

(c)

$$E^2 = \begin{cases} \frac{\rho^2}{9\varepsilon_0^2} r^2, \text{ for } r \le R \\ \frac{\rho^2}{9\varepsilon_0^2}\frac{R^6}{r^4}, \text{ for } r \ge R \end{cases} \quad \text{and} \quad V\mathbf{E} = \frac{\rho R^3}{3\varepsilon_0}\frac{1}{r^2}\frac{\rho R^3}{3\varepsilon_0}\frac{1}{r}\hat{\mathbf{r}} = \frac{\rho^2}{9\varepsilon_0^2}\frac{R^6}{r^3}\hat{\mathbf{r}}$$

$$W = \frac{\varepsilon_0}{2}\left(\int_V E^2 d\tau + \oint_S V\mathbf{E}\cdot d\mathbf{a}\right) = \frac{\varepsilon_0}{2}\left(\int_0^R E^2 d\tau + \int_R^a E^2 d\tau + \oint_{r=a} V\mathbf{E}\cdot d\mathbf{a}\right)$$
$$= \frac{\varepsilon_0}{2}\left(\frac{\rho^2}{9\varepsilon_0^2}\frac{R^5}{5} - \frac{\rho^2}{9\varepsilon_0^2}\left(\frac{R^6}{a} - \frac{R^6}{R}\right)\right)4\pi + \frac{\varepsilon_0}{2}\left(\frac{\rho^2}{9\varepsilon_0^2}\frac{R^6}{a}\right)4\pi$$
$$= \frac{\varepsilon_0}{2}\left(\frac{\rho^2}{9\varepsilon_0^2}\frac{6R^5}{5}\right)4\pi + \frac{\varepsilon_0}{2}\left(\frac{\rho^2}{9\varepsilon_0^2}\left(\frac{R^6}{a} - \frac{R^6}{a}\right)\right)4\pi$$
$$= \frac{4\pi\rho^2 R^5}{15\varepsilon_0}$$

3.

(a)

Simple solution: $\sigma = \frac{Q}{A} = \frac{Q}{4\pi R^2}$

Use Gauss's law, we obtain $E = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$ for $r > R$

Normal component $(E_{above} - E_{below}) = \frac{\sigma}{\varepsilon_0}$, $E_{below} = 0$, $\sigma = \frac{Q}{4\pi R^2}$

(b)

Use Gauss's law, we obtain $E = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$ for $r > R$

$\mathbf{f} = \varepsilon_0 E^2/2\,\hat{\mathbf{n}} = E = \frac{Q^2}{32\pi^2\varepsilon_0}\frac{1}{r^4}\hat{\mathbf{r}}$

Only z-component of $\mathbf{f}$ will survive $f_z = \frac{Q^2}{32\pi^2\varepsilon_0}\frac{1}{R^4}\hat{\mathbf{r}}\cdot\hat{\mathbf{z}} = \frac{Q^2}{32\pi^2\varepsilon_0}\frac{1}{R^4}\cos\theta$

$$F_z = \int_0^{\pi/2}\int_0^{\pi} f_z R^2 \sin\theta d\theta d\phi = \frac{Q^2}{32\pi^2\varepsilon_0}\frac{2\pi}{R^2}\int_0^{\pi/2}\cos\theta\sin\theta d\theta = \frac{Q^2}{32\pi\varepsilon_0}\frac{1}{R^2}$$

4.

(a)

Assume the image line charge of $-\lambda$ is placed at a distance d below the plane.

Using the Gauss's law, the electric field outside a line charge λ is $\mathbf{E} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\hat{\mathbf{r}}$.

So $V = -\int_{r_0}^{r} \mathbf{E}\cdot d\mathbf{l} = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r_0}{r} = V(r) - V_{ref}(r_0)$

$$V = V_+ + V_- = \frac{\lambda}{2\pi\varepsilon_0}\left(\ln\frac{r_0}{\sqrt{(x-d)^2+y^2}} - \ln\frac{r_0}{\sqrt{(x+d)^2+y^2}}\right) = \frac{\lambda}{4\pi\varepsilon_0}\left(\ln\frac{(x+d)^2+y^2}{(x-d)^2+y^2}\right)$$

(b)

$$\sigma = \varepsilon_0\mathbf{E}\cdot\hat{\mathbf{n}} = \varepsilon_0 E_x = -\frac{\partial}{\partial x}\frac{\lambda}{4\pi}\left(\ln\frac{(x+d)^2+y^2}{(x-d)^2+y^2}\right)\Bigg|_{x=0} = -\frac{\lambda}{4\pi}\left(\frac{2(x+d)}{(x+d)^2+y^2} - \frac{2(x-d)}{(x-d)^2+y^2}\right)\Bigg|_{x=0}$$

$$= -\frac{\lambda}{4\pi}\frac{4d}{d^2+y^2} = -\frac{\lambda}{\pi}\frac{d}{d^2+y^2}$$

Simple check: $\lambda' = \int_{-\infty}^{\infty}\sigma dy = \int_{-\infty}^{\infty} -\frac{\lambda}{\pi}\frac{d}{d^2+y^2}dy$

Let $y = d\tan\theta,\ dy = d\sec^2\theta d\theta$

$$\lambda' = -\frac{\lambda}{\pi}\int_{-\pi/2}^{\pi/2}\frac{d^2\sec^2\theta}{d^2\sec^2\theta}d\theta = -\lambda$$

(c)

$dF = Edq = E\lambda d\ell$

$$\frac{dF}{d\ell} = E\lambda = \frac{\lambda}{2\pi\varepsilon_0(2d)}\lambda = \frac{\lambda^2}{4\pi\varepsilon_0 d}$$

5.

(a)

Boundary condition $\begin{cases}\text{(i) } V_1(a,\theta) = V_0(5\cos^3\theta - 3\cos\theta)\\ \text{(ii) } V_2(b,\theta) = 0\\ \text{(ii) } V_3(r,\theta=\pi/2) = 0\end{cases}$

General solution $V(r,\theta) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-(\ell+1)})P_\ell(\cos\theta)$

(b)

B.C. (ii) $\to V(b,\theta)=\sum_{\ell=0}^{\infty}(A_\ell b^\ell + B_\ell b^{-(\ell+1)})P_\ell(\cos\theta)=0 \;\Rightarrow B_\ell = -A_\ell b^{2\ell+1}$

B.C. (i) $\to V(a,\theta)=\sum_{\ell=0}^{\infty}(A_\ell b^\ell + B_\ell b^{-(\ell+1)})P_\ell(\cos\theta)=2V_0P_3(\cos\theta)$

Comparing the coefficiency $\Rightarrow A_3a^3+B_3a^{-4}=2V_0,\; A_\ell=B_\ell=0$ for $\ell=0,1,2,4,5,...$

B.C. (iii) $\to V(r,\theta=\frac{\pi}{2})=(A_3r^3+B_3r^{-3})P_3(0)=0$

$\Rightarrow A_\ell=B_\ell=0$ except $\ell=3$,

$$A_3=\frac{2V_0a^4}{a^7-b^7} \text{ and } B_3=-\frac{2V_0a^4b^7}{a^7-b^7}$$

$$\therefore V(r,\theta)=\left(\frac{V_0}{a^7-b^7}a^4r^3-\frac{V_0}{a^7-b^7}a^4b^7r^{-4}\right)\left(5\cos^3\theta-3\cos\theta\right)$$

(c)

$$V(r,\theta)=\left(\frac{V_0}{a^7-b^7}a^4r^3-\frac{V_0}{a^7-b^7}a^4b^7r^{-4}\right)\left(5\cos^3\theta-3\cos\theta\right)$$

$$\mathbf{E}=-\nabla V=-\frac{\partial V}{\partial r}\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}}$$

$$=-\left(\frac{3V_0}{a^7-b^7}a^4r^2+\frac{4V_0}{a^7-b^7}a^4b^7r^{-5}\right)\left(5\cos^3\theta-3\cos\theta\right)\hat{\mathbf{r}}$$

$$+\left(\frac{V_0}{a^7-b^7}a^4r^3-\frac{V_0}{a^7-b^7}a^4b^7r^{-4}\right)\left(15\cos^2\theta\sin\theta-3\sin\theta\right)\hat{\boldsymbol{\theta}}$$

$$\mathbf{E}(r=a)=-\frac{V_0}{a}\left(\frac{3a^7+4b^7}{a^7-b^7}\right)\left(5\cos^3\theta-3\cos\theta\right)\hat{\mathbf{r}}+V_0\left(\frac{1}{a}\right)\left(15\cos^2\theta\sin\theta-3\sin\theta\right)\hat{\boldsymbol{\theta}}$$

$$\mathbf{E}(r=b)=-V_0\left(\frac{7a^4b^2}{a^7-b^7}\right)\left(5\cos^3\theta-3\cos\theta\right)\hat{\mathbf{r}}$$